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The removability of the Hopf term in the $2+1$D CP^1 model and Aharonov–Bohm effect

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Abstract. In this paper, the removability of the Hopf term in the 2+1D CP^1 model is studied. A quantum unitary transformation is found and used to remove the Hopf term from the Hamiltonian for the model. It is then shown that there is an intimate connection between the Hopf term and the multivaluedness of the wavefunctional describing the model.

1. Introduction

The $2+1$D CP^1 model (or, equivalently, the $2+1$D O(3) nonlinear σ-model) has long attracted considerable attention since it is the first of the field realizations of the speculation of the fractional statistics [1, 2] and may play an important role in the investigation of a number of problems in condensed matter physics [3–6].

It is important to answer the question of whether or not the Hopf term exists in the $2+1$D CP^1 model when it is regarded as the continuum limit of the Heisenberg antiferromagnetic spin model. Fradkin *et al* [7–9] concluded that, as the continuum limit of the spin model, the $2+1$D CP^1 model does not have the Hopf term. Essentially, they drew this conclusion only from the consideration of the relation between the Lagrangians for the spin and the CP^1 models.

In [10], Giavarini *et al* demonstrated the removability of the Berry phase [11] that is, the Berry phase for any one-dimensional system can be removed by a chosen unitary transformation. Later, the idea in [10] was successfully employed to study the removability of the topological term, which is essentially a type of geometrical phase, in the 1+1D CP^1 model [12]. In the present paper, we use the idea in [10] to investigate the removability of the Hopf term in the $2+1$D CP^1 model. A quantum unitary transformation is found and used to remove the Hopf term from the Hamiltonian for the model. It is then shown that there is an intimate connection between the Hopf term and the multivaluedness of the wavefunctional describing the model. The amplitude for the transition from vacuum to vacuum is calculated. From this calculation, it can be seen that the Hopf term gives rise to the geometric phase which is similar to the Berry phase. Finally, we argue that the Hopf term may be the manifestation of the 'molecular' Aharonov–Bohm effect in an infinite-dimensional system with two interacting parts.

The quantization of the $2+1$D CP^1 model is complicated since it is a constrained system with an internal gauge degree of freedom. Recently, Pak and Percacei [13] simplified the quantization by using Euler angles as parameters which remove one constraint automatically [13]. We are to use this parametrization to discuss the removability of the Hopf term.

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2. The classical canonical transformation and the corresponding quantum unitary transformation which removes the Hopf term from the Hamiltonian

The action for the O(3) nonlinear σ-model with the Hopf term is [1, 2]

$$S = \frac{1}{2g^2}\int \mathrm{d}^3x\, \partial_\mu n^a \partial^\mu n^a + \frac{\Theta}{4\pi}\int \mathrm{d}^3x\, \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \qquad n^a n^a = 1,\ a = 1,2,3,\ \mu = 0,1,2 \tag{1}$$

where the gauge potential a_μ is defined through the topologically conserved current $J_\mu = (1/8\pi)\varepsilon^{\mu\nu\lambda}\varepsilon^{abc} n^a \partial_\nu n^b \partial_\lambda n^c = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$ and $\partial_1 = \partial/\partial x^1$, $\partial_2 = \partial/\partial x^2$, $\partial_3 = \partial/\partial t$. The second term in (1) is the Hopf term with coefficient Θ. It is difficult to deal with the action in (1) because of the non-locality of the Hopf term in n^a. The O(3) nonlinear σ-model can be equivalently formulated as the CP^1 model [5] by introducing the variable Z via $n^a = Z^+\sigma^a Z$, where Z is a spinor with two complex components, satisfying the constraint $Z^+Z = 1$, σ^a is the Pauli matrix. Pak and Percacei [13] parametrize Z with Euler angles [13] as follows

$$Z = \begin{pmatrix} \cos(\theta/2)\exp(+\mathrm{i}\varphi/2) \\ \sin(\theta/2)\exp(-\mathrm{i}\varphi/2) \end{pmatrix} \exp(\mathrm{i}\psi/2) \tag{2}$$

leading to the expression for L_Θ

$$L_\Theta = (1/2g^2)(\partial_\mu\theta\partial^\mu\theta + \sin^2\theta\partial_\mu\varphi\partial^\mu\varphi) + (\Theta/16\pi^2)\varepsilon^{\lambda\mu\nu}\sin\theta\partial_\lambda\theta\partial_\mu\varphi\partial_\nu\psi \tag{3}$$

which describes the CP^1 model—a constrained system with ψ being an internal gauge degree. Note that, in the CP^1 model with Pak's parametrization, not only is the Hopf term local in the Euler angles, but the constraint $ZZ^+ = 1$ is removed automatically. This considerably simplifies the quantization procedure for the model.

By means of a canonical procedure, it is easy to obtain the corresponding Hamiltonian [13]:

$$H_\Theta = \int \mathrm{d}^3x[h_\Theta + \lambda(x)C(x)]$$

$$h_\Theta = (g^2/2)[(\pi_\theta - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\varphi\partial_j\psi)^2 + (1/\sin^2\theta)(\pi_\varphi - (\Theta/16\pi^2) \times \sin\theta\varepsilon^{ij}\partial_i\psi\partial_j\theta)] + (1/2g^2)(\partial_i\theta\partial_i\theta + \sin^2\theta\partial_i\varphi\partial_i\varphi) \tag{4}$$

where the sum of the Θ-dependent terms is the Hopf term in the Hamiltonian, $\lambda(x)$ a Lagrange multiplier field, $C(x) = \pi_\psi - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\theta\partial_j\varphi$ a primary constraint. There is no secondary constraint since $C(x)$ commutes with H. With lengthy calculation, it can be shown that the canonical transformation

$$\begin{aligned} &\theta \to \theta' = \theta \qquad \varphi \to \varphi' = \varphi \qquad \psi \to \psi' = \psi \\ &\pi_\theta \to \pi_{\theta'} = \pi_\theta - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\varphi\partial_j\psi \\ &\pi_\varphi \to \pi_{\varphi'} = \pi_\varphi - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\psi\partial_j\theta \\ &\pi_\psi \to \pi_{\psi'} = \pi_\psi - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\theta\partial_j\varphi \end{aligned} \tag{5}$$

transforms h_Θ and L_Θ into $h_{\Theta=0}$ and $L_{\Theta=0}$:

$$\begin{aligned} &h_\Theta \to h_{\Theta=0} = g^2/2(\pi_{\theta'} + \pi_{\varphi'}/\sin^2\theta') + (1/2g^2)(\partial_i\theta'\partial_i\theta' + \sin^2\theta'\partial_i\varphi'\partial_i\varphi') \\ &L_\Theta \to L_{\Theta=0} = \theta'\pi_{\theta'} + \varphi'\pi_{\varphi'} + \psi'\pi_{\psi'} - h_0' = (1/2g^2)(\partial_\mu\theta'\partial_\mu\theta' + \sin^2\theta'\partial_\mu\varphi'\partial_\mu\varphi'). \end{aligned} \tag{6}$$

$h_{\Theta=0}$ and $L_{\Theta=0}$ no longer contain the Hopf term which has been removed by the canonical transformation in (5). Suppose that $L_{\Theta=0}$ (in which the Hopf term is absent) is that obtained from the Lagrangian for the spin model by taking the continuum limit, the inverse of the

canonical transformation (5) can be used to transform $L_{\Theta=0}$ into L_Θ in which there is the Hopf term. This means that the Hopf term is only a surface term. Thus, in order to study the physical meaning of the Hopf term in the CP^1 model, it appears to be necessary to quantize the model and construct the unitary transformation corresponding to the classical canonical transformation (5).

The quantum Hamiltonian is [13]

$$\begin{aligned} H_\Theta &= \int \mathrm{d}^2x\,[h_\Theta + \lambda(x)C(x)] \\ h_\Theta &= g^2/2\{(1/\sin\theta)[\pi_\theta - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\varphi\partial_j\psi] \\ &\quad \times \sin\theta[\pi_\theta - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\varphi\partial_j\psi] \\ &\quad + \sin^2\theta(\pi_\varphi - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\psi\partial_j\theta)^2\} \\ &\quad + (1/2g^2)(\partial_i\theta\partial_i\theta + \sin^2\theta\partial_i\varphi\partial_i\varphi) \end{aligned} \tag{7}$$

where $C(x) = \pi_\psi - (\Theta/16\pi^2)\sin\theta\varepsilon^{ij}\partial_i\theta\partial_j\varphi$ is the constraint operator. As usual, the commutation relations are

$$\begin{aligned} &[\theta(x), \pi_\theta(x')] = \mathrm{i}\delta^2(x - x') \\ &[\varphi(x), \pi_\varphi(x')] = \mathrm{i}\delta^2(x - x') \\ &[\psi(x), \pi_\psi(x')] = \mathrm{i}\delta^2(x - x') \end{aligned} \tag{8}$$

and the physical state $|\Psi\rangle_{\mathrm{phy}}$ satisfies $C(x)|\Psi\rangle_{\mathrm{phy}} = 0$. With considerable effort, we find the global expression for the quantum unitary transformation

$$U = \exp\left[\mathrm{i}(\Theta/16\pi^2)\int_{\mathrm{c}} \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}\tau\;\varepsilon^{\mu\nu\lambda}\sin\theta\partial_\mu\theta\partial_\nu\varphi\partial_\lambda\psi\right] \tag{9}$$

where C represents the path (which is parametrized by τ) and $\partial_1 = \partial/\partial x^1$, $\partial_2 = \partial/\partial x^2$, $\partial_3 = \partial/\partial\tau$. The path C starts from a reference configuration $\{\theta_0(x), \varphi_0(x), \psi_0(x)\}$ at $\tau = \tau_i$ and ends at the configuration $\{\theta(x), \varphi(x), \psi(x)\}$ at $\tau = \tau_f$ where the local calculation for $UH_\Theta U^+$ is to be carried out. With the help of the fact that $\varepsilon^{\mu\nu\lambda}\sin\theta\partial_\mu\theta\partial_\nu\varphi\partial_\lambda\psi = \partial_\mu\omega^\mu$ with $\omega^\mu = \varepsilon^{\mu\nu\lambda}\psi\sin\theta\partial_\nu\theta\partial_\lambda\varphi$ is a total derivative, we obtain for the integral in (9)

$$\int_c \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}\tau\;\partial_\mu\omega^\mu = \int \mathrm{d}x^1\,\mathrm{d}x^2\,[\omega(\theta, \varphi, \psi) - \omega(\theta_0, \varphi_0, \psi_0)] \tag{10}$$

by noting that, as usual, there is no contribution from the spatial infinity. It is then easy to carry out the local calculation for $UH_\Theta U^+$ to get

$$UH_\Theta U^+ = H_{\Theta=0}. \tag{11}$$

This means that the Hopf term can be removed by U which corresponds to the classical canonical transformation in (5). Note that, in order to globally define the unitary transformation, one must employ (9). This globally defined U is apparently multivalued.

Now, we turn to the discussion of the physical meaning of the Hopf term. According to the basic principle of the quantum field theory, the U transforms the Hamiltonian H_Θ and the physical state vector $|\Psi\rangle_{\mathrm{phy}}$, which is single-valued, in the following way

$$H_\Theta \to UH_\Theta U^+ = H_{\Theta=0} \tag{12}$$

$$|\Psi\rangle_{\mathrm{phy}} \to U|\Psi\rangle_{\mathrm{phy}} = |\Psi, \Theta\rangle_{\mathrm{phy}}. \tag{13}$$

Apparently $|\Psi, \Theta\rangle_{\mathrm{phy}}$ is multivalued since U is multivalued. This means that the CP^1 model described by the single-valued wavefunctional with the Hamiltonian H_Θ (in which the Hopf term is present) is physically equivalent to the CP^1 model described by the multivalued

wavefunctional with the Hamiltonian $H_{\Theta=0}$ (in which the Hopf term is absent). This indicates that the Hopf term actually represents a type of Aharonov–Bohm effect in the CP^1 model—an infinite-dimensional system.

In quantum field theory, it is important to calculate the amplitude for the transition from vacuum to vacuum. Because of the stability of the vacuum state, the cyclic condition is satisfied and the amplitude is actually the phase factor in which there is a geometric phase similar to the Berry phase. With the help of the complete set of the shape states $|\theta(x), \varphi(x), \psi(x)\rangle$, we calculate the amplitude $\langle 0|\exp(-H_\Theta T)|0\rangle$ and obtain

$$\begin{aligned}
\langle 0|\exp(-H_\Theta T)|0\rangle &= C_N \int \left(\prod_{m=0}^{N-1}\prod_{x_1x_2} \mathrm{d}p_m\,\mathrm{d}k_m\,\mathrm{d}q_m\right)\left(\prod_{n=1}^{N-1}\prod_{x_1x_2}\sin\theta_n\,\mathrm{d}\theta_n\,\mathrm{d}\varphi_n\,\mathrm{d}\psi_n\right)\\
&\quad\times\exp\left\{\varepsilon\sum_{k=0}^{N-1}\int \mathrm{d}^2x\,(p_k\theta_k + k_k\sin\theta_k\varphi_k + q_k\psi_k)\right.\\
&\quad -\frac{g^2}{2}\left[\left(p_k - \frac{\Theta}{16\pi^2}\sin\theta_k\varepsilon^{ij}\partial_i\varphi_k\partial_j\psi_k\right)^2\right.\\
&\quad \left.+\frac{1}{\sin^2\theta_k}\left(k_k - \frac{\Theta}{16\pi^2}\sin\theta_k\varepsilon^{ij}\partial_i\psi_k\partial_j\theta_k\right)^2\right]\\
&\quad \left.+\frac{1}{2g^2}[(\partial_i\theta_k)^2 + \sin^2\theta_k(\partial_i\varphi_k)^2] + \mathrm{i}\lambda_k\left(q_k - \frac{\Theta}{16\pi^2}\sin\theta_k\varepsilon^{ij}\partial_i\theta_k\partial_j\varphi_k\right)\right\}\\
&= C_N\int\left(\prod_{x_0x_1x_2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\psi\right)\\
&\quad\times\exp\left\{-\varepsilon\int\mathrm{d}^3x\left[\frac{1}{2g^2}(\partial_\mu\theta\partial^\mu\theta + \sin^2\theta\partial_\mu\varphi\partial^\mu\varphi)\right.\right.\\
&\quad \left.\left.+\mathrm{i}\frac{\Theta}{16\pi^2}\varepsilon^{\lambda\mu\nu}\sin\theta\partial_\lambda\theta\partial_\mu\varphi\partial_\nu\psi\right]\right\} \qquad (14)
\end{aligned}$$

where $T = N\varepsilon(N\to\infty)$, ε is infinitesimally small. Note that the constraint has been considered via the integration over the Lagrange multiplier field λ. The term $\mathrm{i}\frac{\Theta}{16\pi^2}\varepsilon^{\lambda\mu\nu}\sin\theta\partial_\lambda\theta\partial_\mu\varphi\partial_\nu\psi$ in (14) is the Hopf term and gives rise to the geometric phase factor.

3. Discussion

A molecule is a system which contains two interacting parts—a nuclear and an electron part. When the Born–Oppenheimer method is used to study a molecule, the effective Hamiltonian for the nuclear part will contain a Berry connection term [14], as both the wavefunctions for the nuclear and electron parts are required to be single-valued. If the wavefunction for the electron part as well as that for the nuclear part are allowed to be multivalued (the product of them remains single-valued), the Berry connection term can be removed. This is referred to as the molecular Aharonov–Bohm effect [15]. The Hopf term may represent a type of 'molecular' Aharonov–Bohm effect in an infinite-dimensional system which contains two interacting parts of which one is the CP^1 (or spin) system. If this is the case, one has to find another part which interacts with the CP^1 (or spin) system. Work in this direction is under investigation.

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